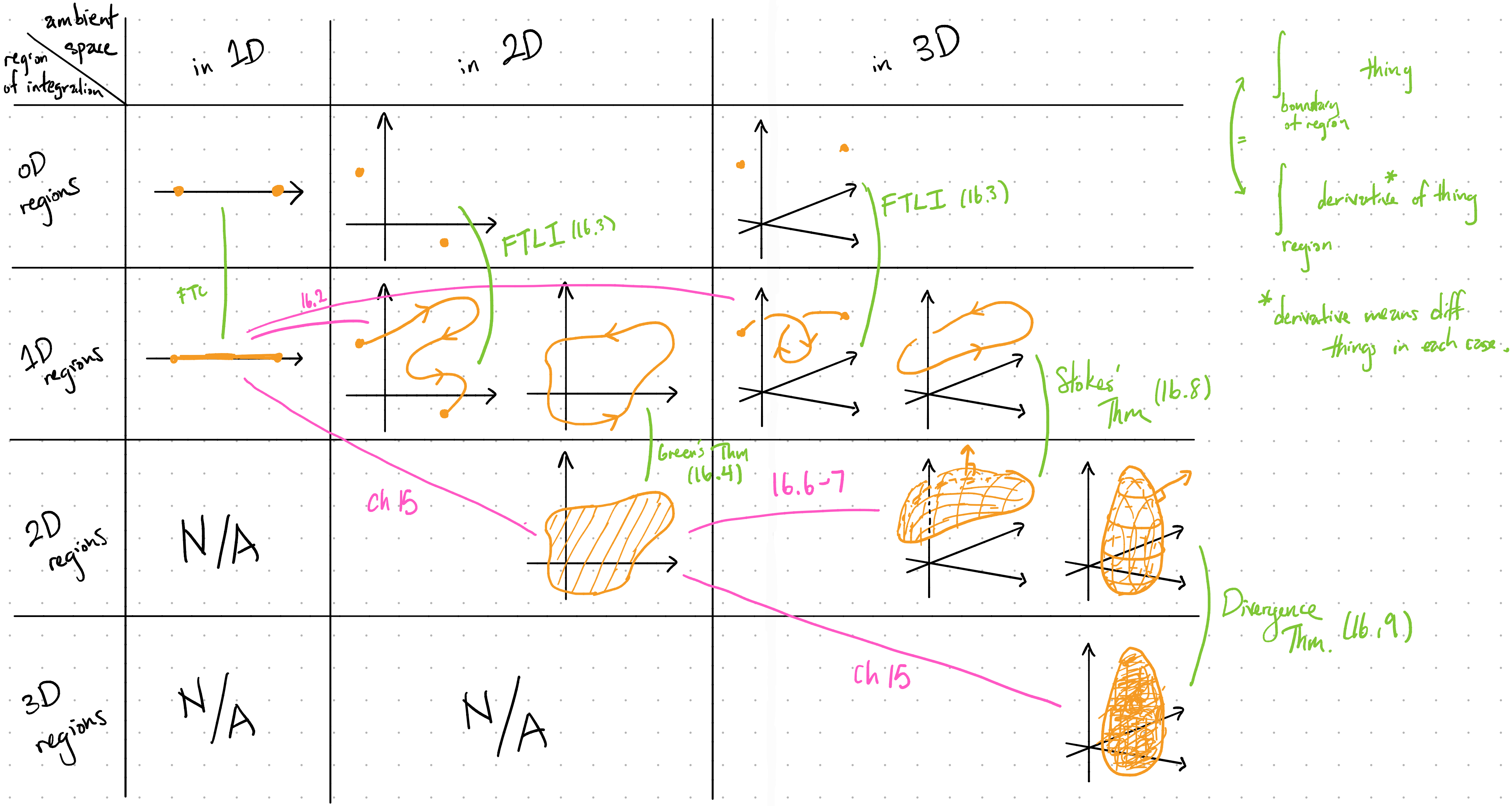
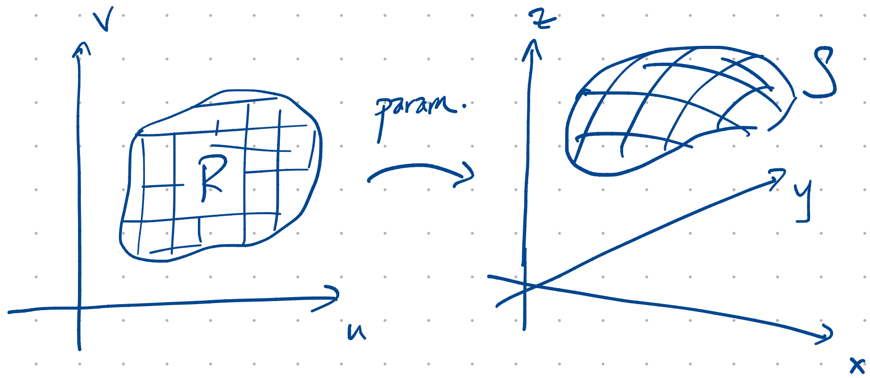

ambient space
region of integration
in 1D
in 2D
in 3D
0D regions
1D regions
2D regions
3D regions
FTC
FTLI (16.3)
FTLI (16.3)
16.2
Green's Thm (16.4)
Stokes' Thm (16.8)
16.6-7
Ch 15
Ch 15
N/A
N/A
N/A
Divergence Thm. (16.9)
∫ boundary of region thing
=
∫ region derivative* of thing
* derivative means diff things in each case.

If $\vec{r}(u,v) = \langle x,y,z \rangle$ $(u,v) \in R \leftarrow$ u,v region

parametrizes a surface S



then

$$\iint_S f(x,y,z)\, dS = \iint_R f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

↑ substitute in terms of u,v

where $\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$

$\vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$

Let's consider 2 special cases:

(1.) the surface is the graph of some function $f(x,y)$.

i.e.

S: $z = f(x,y)$.

and we use the "natural" parametrization with x, y as parameters:

$x = x$
$y = y$
$z = f(x,y)$

Then:

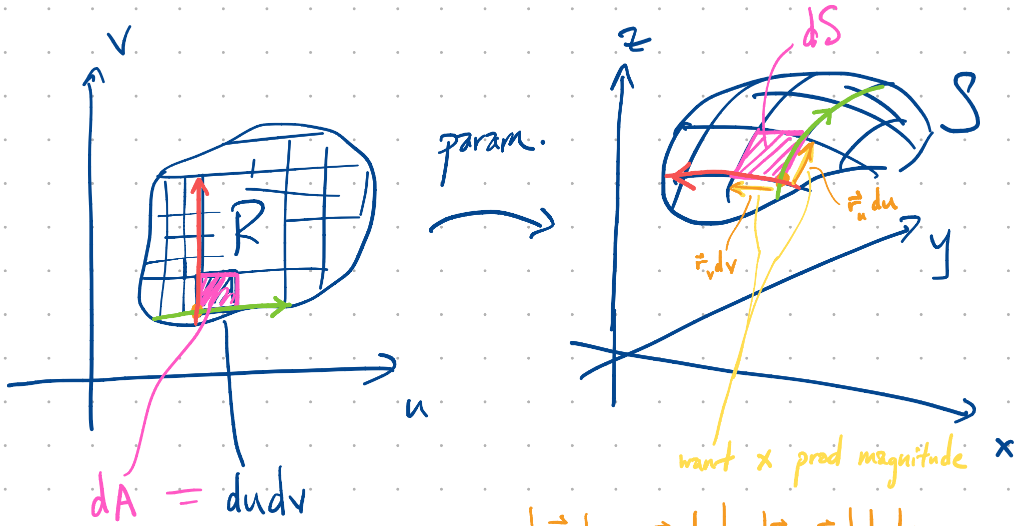
$\vec{r}_x = \langle 1, 0, f_x(x,y) \rangle$

$\vec{r}_y = \langle 0, 1, f_y(x,y) \rangle$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1 \rangle$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{f_x^2 + f_y^2 + 1}$$

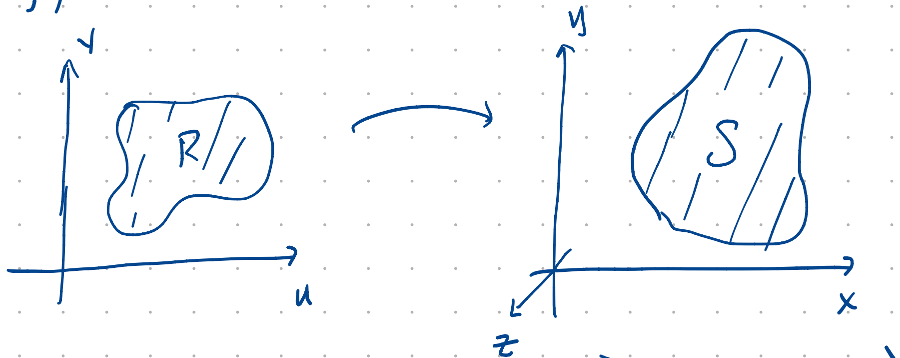
i.e. we recover the formula from 15.5



$$\left|\vec{r}_u \, du \times \vec{r}_v \, dv\right| = \left|\vec{r}_u \times \vec{r}_v\right| du\,dv$$

"scalar"

(2) The surface S actually resides entirely in the xy-plane:



$\vec{r}(u,v) = \langle x, y, 0 \rangle$

$$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, 0 \right\rangle$$

$$\vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, 0 \right\rangle$$

$$\vec{r}_u \times \vec{r}_v = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{bmatrix} = \left\langle 0, 0, \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \left| \ldots\ldots \right| = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

so we recover the formula from 15.9

So 15.5 and 15.9 (in 2D) are both special cases of 16.6.

Rmk: $\nabla \times \langle P, Q \rangle$ doesn't technically make sense*

but

$$\nabla \times \langle P, Q, 0 \rangle = \langle 0, 0, Q_x - P_y \rangle$$

*b/c curl only def for 3D vec fields